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Some Characterizations of Strange Sets

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A thermodynamic formalism is exhibited that is the canonical version of Halsey *et al.*'s microcanonical formulation. This formalism is applied to a four-scale Cantor set and it is shown that the singularity spectrum fails to uniquely encode the underlying dynamics.

KEY WORDS: Spectrum of singularities; thermodynamic formalism; transfer matrix; scaling function; dynamical systems.

1. INTRODUCTION

Halsey *et al.*[1] have introduced a method of extracting a spectrum of scalings from experimentally or numerically generated strange sets. We shall show that this method is the microcanonical version of the canonical thermodynamic formalism of Ruelle *et al.*[2] Hereafter we shall refer to Halsey *et al.* as MP (microcanonical paradigm), and to Ruelle *et al.* as CP (canonical paradigm).

Our presentation consists in constructing the canonical ensemble in a form suitable for theoretical calculations on a dynamical system, relating it to the microcanonical formulation, and producing a conversion dictionary between the standard CP vocabulary and the MP functions as defined in Ref. 1. We then demonstrate that in the simplest nontrivial example, the Cantor set characterized by four scales, MP depends only upon three independent combinations of scales, and thus fails to uniquely characterize the set.

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2. THE SINGULARITY SPECTRUM ACCORDING TO CP

Consider a dynamical system whose attractor can be hierarchically presented as a set of N_n intervals $I_k^{(n)}$, $k = 1,..., N_n$, of lengths $\Delta_k^{(n)}$ at the nth level.

Recipe. Construct the canonical "free energy" according to the definition

$$N_n^{-F(\beta)} = \sum_k |\Delta_k^{(n)}|^\beta \tag{2.1}$$

where we understand that asymptotically in n, F becomes independent of n. The relation of the MP functions defined by Halsey *et al.* to the CP quantities is given by the following dictionary:

$$\alpha = 1/F'(\beta) \tag{2.2}$$

$$f = \beta - F(\beta)/F'(\beta) \tag{2.3}$$

$$q = -F(\beta) \tag{2.4}$$

$$\tau = -\beta \tag{2.5}$$

$$D_q = \beta/[1 + F(\beta)] \tag{2.6}$$

By (2.2) and (2.3) we mean that $f(\alpha)$ implicitly parametrized by β *is* $f(\alpha)$ of MP. By (2.4) and (2.5), a plot of q versus $\tau(q)$ is precisely $F(\beta)$. Since q versus τ *is* what is experimentally available, and F is the natural theoretical object, one could probably dispense with f and use (2.2) to talk about the "range of scales."

Observe in (2.1) that, since $|\Delta| < 1$, $F(\beta)$ is a monotonic increasing function diverging with β at $\pm\infty$. It has a unique zero, which is an upper bound to, or the Hausdorff dimension itself. By (2.4) and (2.6) this is D_0. By (2.2) and (2.6), $D_{\pm\infty} = \alpha_{\pm\infty}$.

We shall show that (2.2) and (2.3) are the quantities defined in MP, while (2.4)–(2.6) are internal consequences of MP. However, first we shall explore CP with two examples.

3. EXAMPLES

3.1. Two-Scale Cantor Set

Consider a Cantor set whose nth level consists of intervals

$$\Delta^{(n)}(\varepsilon_n,..., \varepsilon_1) = \sigma_1^n(\sigma_2/\sigma_1)^{\Sigma_1^n \varepsilon}, \qquad \varepsilon_i = 0, 1 \tag{3.1}$$

$$N_n = 2^n \tag{3.2}$$

Observe that

$$\frac{\Delta^{(n)}(\varepsilon_n, \varepsilon_{n-1},..., \varepsilon_1)}{\Delta^{(n-1)}(\varepsilon_{n-1},..., \varepsilon_1)} = \sigma(\varepsilon_n) = \sigma_1^{1-\varepsilon_n}\sigma_2^{\varepsilon_n} \tag{3.3}$$

Denoting the number of iterates of the critical point by $t = \varepsilon_1 + \cdots + 2^{n-1}\varepsilon_n$, we have that $\varepsilon_n = 0$ denotes the first quarter of t through N_{n+1}, and $\varepsilon_n = 1$ the second. Thus, σ_1 and σ_2 represent the leading two-scale approximation to the scaling function of Ref. 3, hereafter referred to as SP (the scaling paradigm).

By (2.1),

$$\begin{aligned} e^{-nF(\beta)\ln 2} &= \sigma_1^{\beta n} \sum_{\{\varepsilon_1,...,\varepsilon_n\}} (\sigma_2/\sigma_1)^{\beta\sum_1^n \varepsilon} \\ &= \sigma_1^{\beta n}\left[\sum_{\varepsilon=0}^{1} (\sigma_2/\sigma_1)^{\beta\varepsilon}\right]^n \\ &= (\sigma_1^\beta + \sigma_2^\beta)^n \end{aligned}$$

or

$$F(\beta) = -\ln(\sigma_1^\beta + \sigma_2^\beta)/\ln 2 \tag{3.4}$$

Equation (3.4) together with (2.2) and (2.3) reproduces the graph of $f(\alpha)$ of MP.

3.2. Four Scale Contor Set

Consider a Cantor set whose nth level consists of intervals characterized by four scales

$$\begin{aligned} &\Delta^{(n)}(\varepsilon_n,..., \varepsilon_1) \\ &\quad = \sigma_1^n(\sigma_2/\sigma_1)^{\sum_1^n \varepsilon}(\sigma_3/\sigma_1)^{\sum_1^{n-1}\varepsilon}(\sigma_1\sigma_4/\sigma_2\sigma_3)^{\sum_1^{n-1}\varepsilon_i\varepsilon_{i+1}}, \qquad \varepsilon_i = 0, 1 \end{aligned} \tag{3.5}$$

As in (3.3),

$$(\Delta^{(n)}/\Delta^{(n-1)})^\beta = \sigma^\beta(\varepsilon_n, \varepsilon_{n-1}) \equiv \sigma^\beta(\varepsilon, \varepsilon') = T_{\varepsilon'\varepsilon}$$

where

$$T = \begin{pmatrix} \sigma_1^\beta & \sigma_2^\beta \\ \sigma_3^\beta & \sigma_4^\beta \end{pmatrix} \tag{3.6}$$

$\sigma_1,..., \sigma_4$ is just the leading four-scale approximation to the SP scaling function $\sigma(t)$, with σ taken as constant at σ_i on intervals of 1/8 of N_{n+1}.

We see by (3.6) that T, i.e., the SP scaling function, is simply the transfer matrix of the Ising model that (3.5) produces in canonical ensemble when substituted in (2.1).

Denoting the larger eigenvalue of T by $\lambda(\beta)$, it follows from (2.1) that

$$F(\beta) = -\ln \lambda(\beta)/\ln 2 \tag{3.7}$$

where

$$\lambda(\beta) = \frac{\sigma_1^\beta + \sigma_4^\beta}{2} + \left[\left(\frac{\sigma_1^\beta - \sigma_4^\beta}{2}\right)^2 + (\sigma_2\sigma_3)^\beta\right]^{1/2} \tag{3.8}$$

Thus, the theoretical $f(\alpha)$, D_q or whatever, is now available for this four-scale Cantor set. The reader *must* realize that period doubling dynamics *does* produce four measurable scales. The thermodynamic quanity F, however, depends on σ_2 and σ_3 only through the *combination* $\sigma_2\sigma_3$.

That is, the MP description $f(\alpha)$ of scalings of strange sets other than the trivial two-scale Cantor set is *infinitely degenerate* over internal scales: already in the four-scale approximation to the period doubling attractor, MP fails to distinguish the period doubling attractor from the one-dimensional family of other strange sets with the same product $\sigma_2\sigma_3$. None of this massaging of experimental data can justify the claim that a definite dynamics qua metric has been observed.

4. THE CONNECTION OF CP TO MP

Starting with (2.1), write

$$N_n^{-F(\beta)} = \sum_\mu N^{-\beta\mu}\left(\sum_{\substack{k \\ h(m_1,\ldots,m_l)=\mu}}\right) \tag{4.1}$$

where we have defined

$$-\ln \Delta_k^{(n)}/\ln N_n \underset{n\to\infty}{\sim} h(m_1,\ldots, m_l) \tag{4.2}$$

with $m_1,\ldots, m_l$ a sufficient set of *intensive* variables that label the kth interval $I_k^{(n)}$. We expect in (4.2) that $\Delta_k^{(n)}$ and N_n have exponential dependence on n. The relation (4.2) defines h to be the Hamiltonian per "site" n. In the example (3.5) we see that h is an Ising Hamiltonian, and requires two intensive parameters

$$\frac{1}{n}\sum_1^n \varepsilon_i \qquad \text{and} \qquad \frac{1}{n}\sum_1^{n-1} \varepsilon_i\varepsilon_{i+1}$$

to specify a definite value of h, and so Δ.

Consider the microcanonical ensemble in (4.1):

$$\left(\sum_{\substack{k \\ h(m)=\mu}} \right) \sim N_n^{s(\mu)} \tag{4.3}$$

defining the microcanonical entropy $s(\mu)$. Thus, (4.1) becomes

$$N_n^{-F(\beta)} = \sum_{\mu} N^{s(\mu)-\beta\mu} \tag{4.4}$$

The microcanonical version of F is then

$$F = \beta\bar{\mu} - s(\bar{\mu}), \qquad \text{where} \quad \beta = s'(\bar{\mu}) \tag{4.5}$$

determines the maximizing $\bar{\mu}(\beta)$. Properties (2.2) and (2.3) now follow from the identifications

$$\alpha = 1/\bar{\mu}, \qquad f = s(\bar{\mu})/\bar{\mu} \tag{4.6}$$

and Legendre transformation.

It is now easy to see that α and f of (4.6) are just those of MP. At $h=\bar{\mu}$, the length of an interval by (4.2) is $l = N_n^{-\bar{\mu}}$. Since each piece is visited with probability N_n^{-1}, this is $p(l) = l^{1/\bar{\mu}} \equiv l^{\alpha}$. The microcanonical sum in (4.3) is the number of intervals with $\alpha = 1/\bar{\mu}$, which in MP is $l^{-f(\alpha)}$. This verifies the second part of (4.6). Thus, MP = CP.

5. CONCLUSION

In practice, MP is a convenient tool for extracting three numbers characterizing a strange set: the Hausdorff dimension, the minimal scaling, and the maximal scaling. While we, too, believe, along with Halsey *et al.*, that MP represents a significant breakthrough in numerical analysis of strange sets, MP encodes only a small part of the metrical structure of the underlying dynamics.

The standard thermodynamic CP formalism is the correct machine for the theoretical detrmination of MP quantities; formula (3.8) would have been very hard to come by if computed in MP. CP makes it very clear that a *full* metric invariant, such as the SP scaling function $\sigma(t)$, is the necessary theoretical ingredient for CP to transform into MP. Whether or not σ is ugly and perverse turns on one's viewpoint. Since σ is defined only on the Cantor set—which is all that we are describing—it is perhaps worth noting that σ here is not only continuous, but quite differentiable on the strange set S.

In this connection Ref. 4 might also be of interest.

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